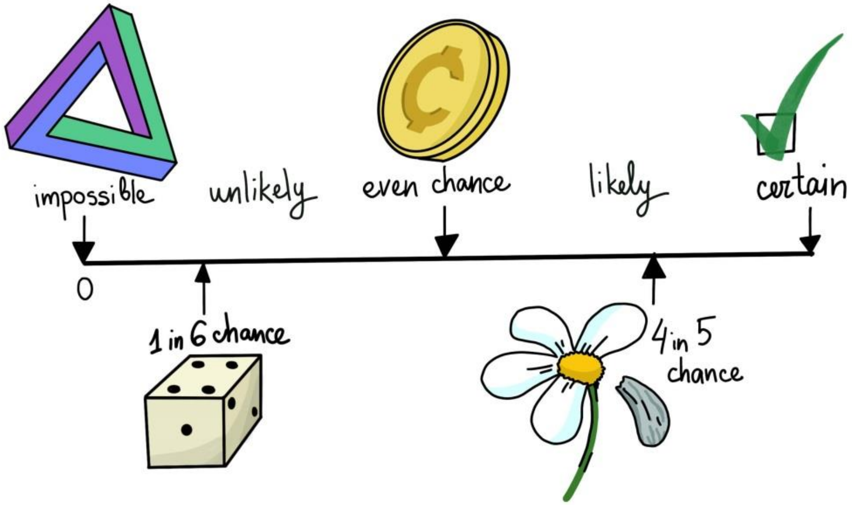
Introduction to Probability



Prepared and Presented by:

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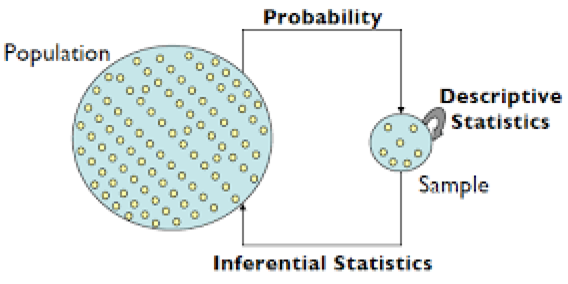
Introduction to Probability

- **Experiments, Outcomes, Events and Sample Spaces**
- **What is probability?**
- **Basic Rules of Probability**
- **Probabilities of Compound Events**

Introduction to Probability

Why Learn Probability?

- Nothing in life is certain. In everything we do, we gauge the chances of successful outcomes, from business to medicine to weather.
- A probability provides a quantitative description of chances or likelihoods associated with various outcomes.
- It provides a bridge between descriptive and inferential statistics.



Basic Probability Concepts

Experiment

- An **experiment** is the process by which an observation (or measurement) is obtained.

- **Deterministic** Vs **non-Deterministic experiment**

➢ Deterministic or Non-Random experiment

➢ Non-Deterministic or Random experiment

Experiment

- **Deterministic** or **Non**-Random experiment
- In **deterministic** experiment, the outcome can be predicted exactly in advance by using a mathematical mode that allows a perfect prediction of the phenomena's outcome.

- Many examples exist in physics and science.
- Example: Force = mass* acceleration. So if we are given values for mass and acceleration, we exactly know the value of force.

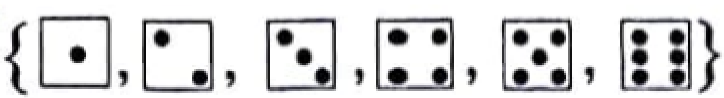
Experiment (Phenomena)

- **Non**-Deterministic or **Random** experiment
- In random experiment, no mathematical model exists that allows a perfect prediction of the experiment's outcome.
- In this case, we are unable to predict the outcomes, or they are not known exactly. However, in the long-run, the outcomes exhibit a statistical regularity, so we can describe the probability of the possible outcomes.

Experiment (Phenomena)

- **Examples**:

1. Tossing a coin, the outcomes S={**Head**, **Tail**}. In this case, we don't know exactly what we get, but in the long run we can calculate the probability and predict that 50% of the time we will get a head and 50% of the time we will get a tail.
2. Rolling a die, so the outcomes

   S=

- We are unable to predict an outcome, but in the long run, we can determine that each outcome will occur 1/6 of the time.

The Sample Space (S)

- The sample space (S or Ω) for a random phenomena (random experiment) is the set of all possible outcomes.

- The sample space S may contain:

1. A finite number of outcomes
2. A countably infinite number of outcomes, or
3. An uncountably infinite number of outcomes.


MORE EXAMPLES
PLEASE

The Sample Space (S)

- **Examples of a sample space (S)**

1. Tossing a coin. Outcomes, S = {Head, Tail} = {H, T}
2. Tossing two coins once. Outcomes S = {HH, HT, TH, TT}. So number of outcomes = n(S) = 2*2= 4.
3. Tossing three coins once. Then, outcomes:

   S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

   Number of outcomes = n(S) = 2*2*2 = 8 or S^n .
4. Rolling a die, then outcomes S = {1, 2, 3, 4, 5, 6},

   So number of outcomes = n(S) = 1*6 = 6.

The Sample Space (S)

5. Rolling two dice, the outcomes are:

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

So number of outcomes S = n (S) = 6*6 = 36

The *mn* Rule

- If an experiment is performed in two stages, with *m* ways to accomplish the first stage and *n* ways to accomplish the second stage, then there are *mn* ways to accomplish the experiment.
- This rule is easily extended to *k* stages, with the number of ways equal to

$$n_1 n_2 n_3 \ldots n_k$$

Example: Toss two coins. The total number of simple events is:

$$2 \times 2 = 4$$

Examples

Example: Toss three coins. The total number of simple events is:

$$2 \times 2 \times 2 = 8$$

Example: Toss two dice. The total number of simple events is:

$$6 \times 6 = 36$$

Example: Two M&Ms are drawn from a dish containing two red and two blue candies. The total number of simple events is:

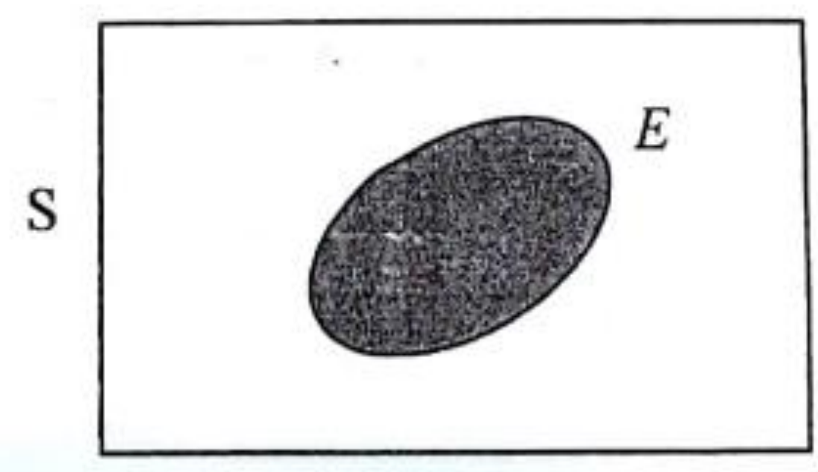
$$4 \times 3 = 12$$

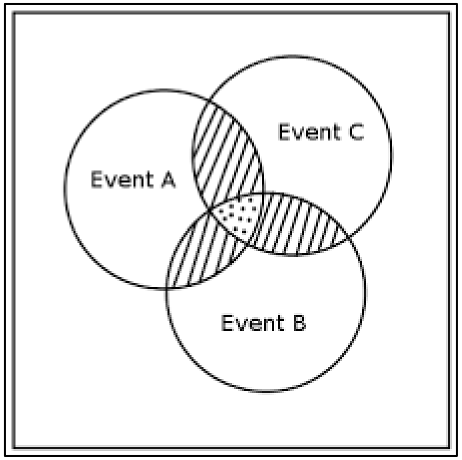
The Event (E)

An Event, E

- The event, which is denoted by E, is any subset of the sample space, S, so it is any set of outcomes (not necessarily all outcomes) of the random phenomena.
- The event, E, can be denoted by A, B, C, D,....

Venn diagram





Types of Events

- There are four types of events as follows:

1. **Null (impossible) event (ϕ)**
2. **Entire (sure or certain) event (S)**
3. **Simple event**
4. **Compound event**

Types of Events

1. The **null** event (the empty event) (ϕ), never occurs, impossible.

   ϕ = { } = the event that contains no outcomes

2. The **entire** (sure or certain) event, the sample space (S), S = the event that contains all outcomes

   So the sure event, S, always occurs.

Types of Events

3. A **simple** event is the outcome that is observed on a single repetition of the experiment.

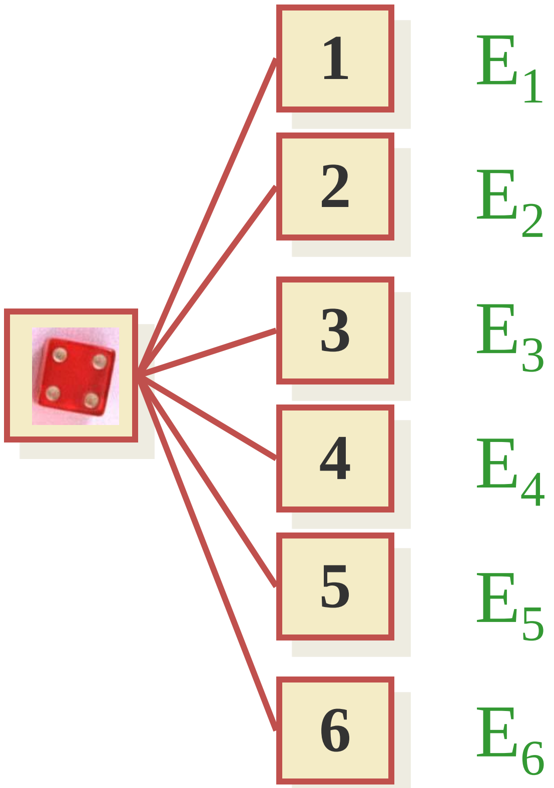
- One and only one simple event can occur when the experiment is performed.
- Simple event is denoted by E with subscript, e.g. E_1, E_2, ..etc.

> **The set of all simple events of an experiment is called the sample space, S.**

4. Compound event is the outcome that contains more than one simple event when the experiment is performed.

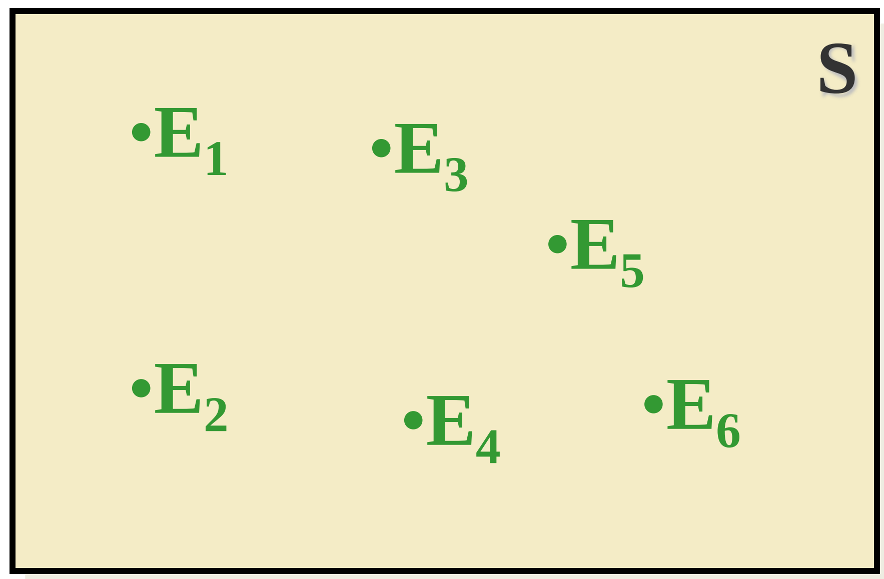
Example

- **The die toss:**
- Simple events:



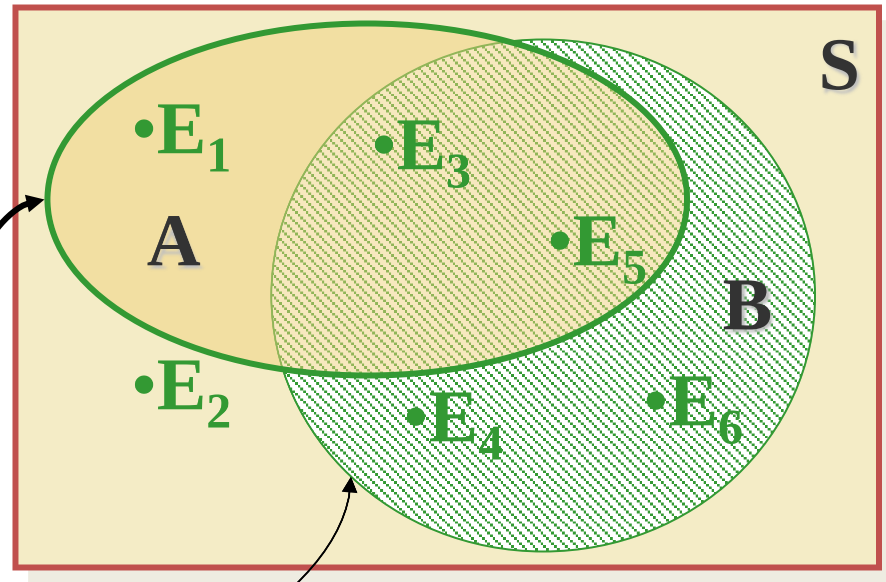
Sample space:

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6\}$$



Event

- An **event** is a collection of one or more **simple events.**



- **The die toss:**
  - –A: an odd number
  - –B: a number > 2

A =$\{E_1, E_3, E_5\}$

B =$\{E_3, E_4, E_5, E_6\}$

Examples of Event

- Rolling a die, then outcomes, S = {1, 2, 3, 4, 5, 6}

➢ **A** = The event that the number comes up is greater than 5 = {6} which is simple event.

➢ **B** = The event that the number comes up is an even number = {2, 4, 6}, which is a compound event.

➢ **C** = Then event that the number comes is less than 7 = {1, 2, 3, 4, 5, 6}, which is a sure event.

➢ **D** = The event that the number comes up is greater than 6 = { } = ϕ, which is an impossible event.

Event Relations

Some Basic Relationships of Probability

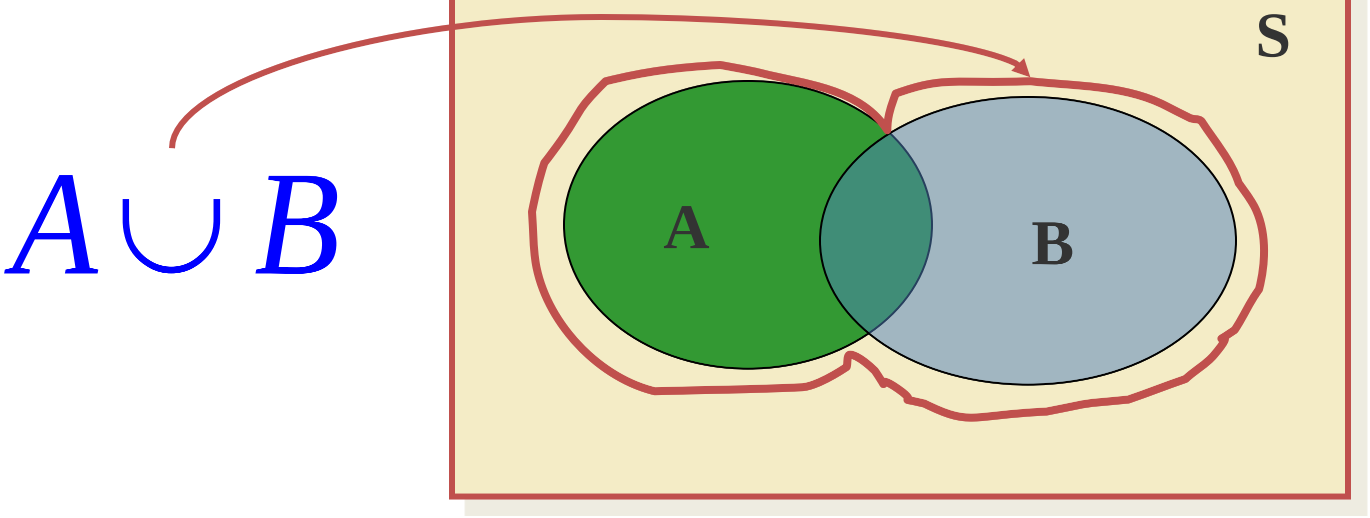
There are some basic probability relationships that can be used to compute the probability of an event without knowledge of all the sample point probabilities.

- Union of Two Events
- Intersection of Two Events
- Complement of an Event
- Mutually Exclusive Events

Union of Two Events

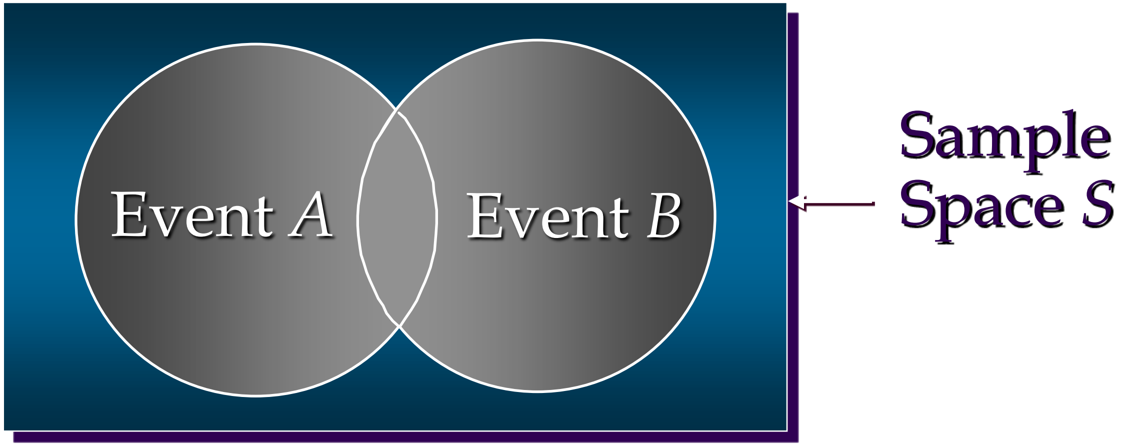
- The **union** of two events, A and B, is the event that either A **or** B **or both** occur when the experiment is performed. We write

$$A \cup B$$



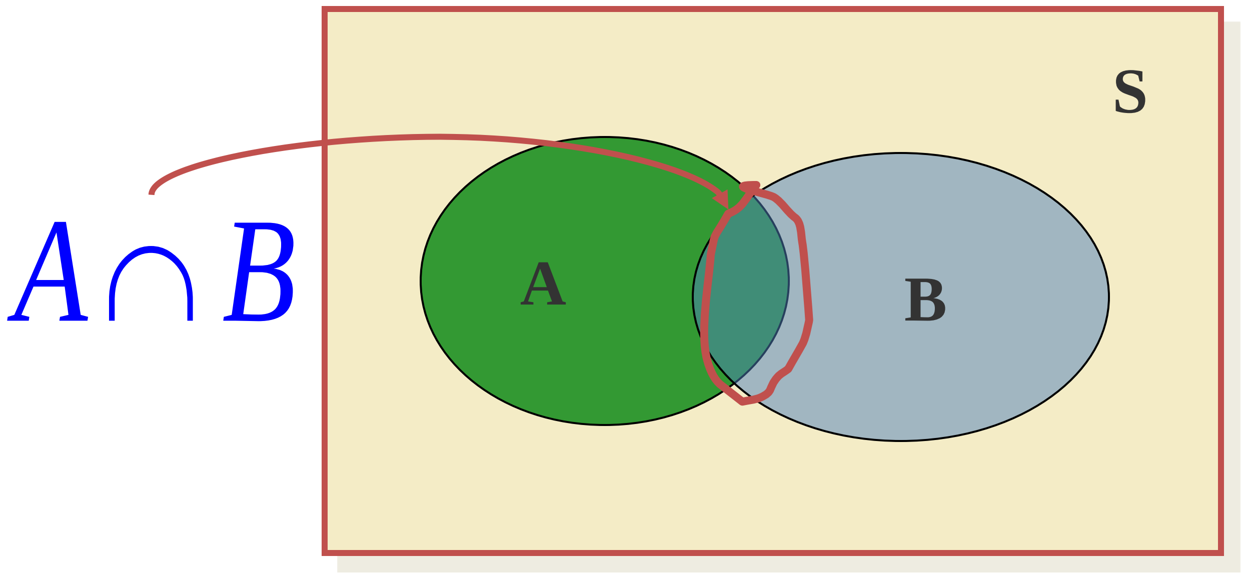
Union of Two Events

- The union of events A and B is the event containing all sample points that are in A or B or both.
- The union of events A and B is denoted by $A \cup B$.



Intersection of Two Events

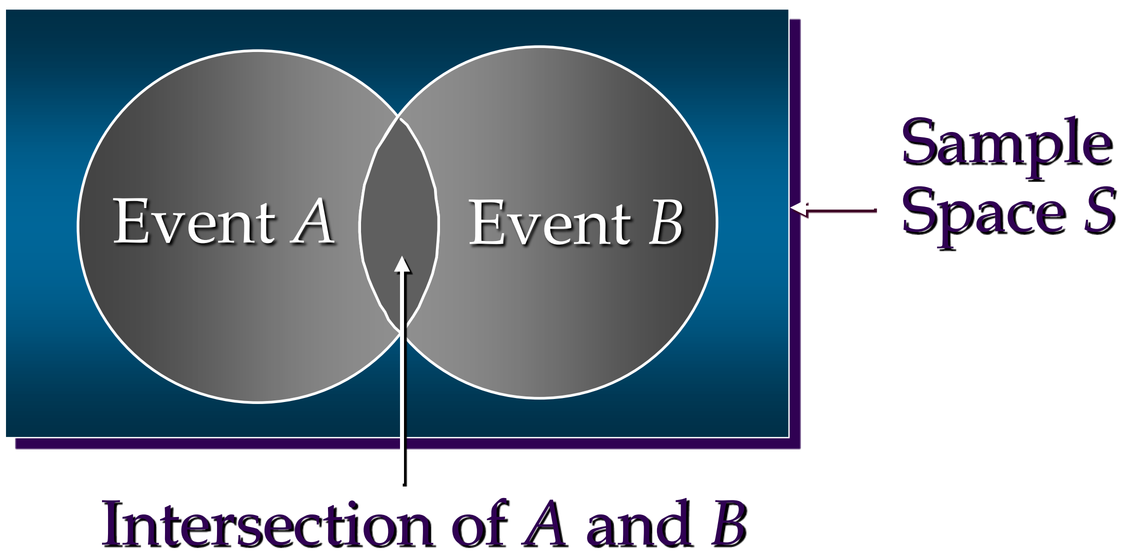
- The **intersection** of two events, **A** and **B**, the event that both A **and** B occur when the experiment is performed. We write **A ∩ B.**



- If two events A and B are **mutually exclusive,** then P(**A ∩ B) = 0.**

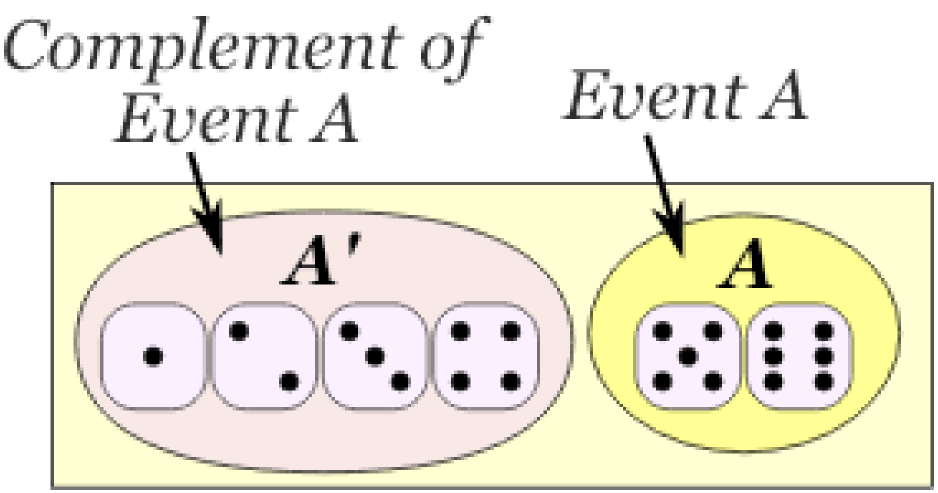
Intersection of Two Events

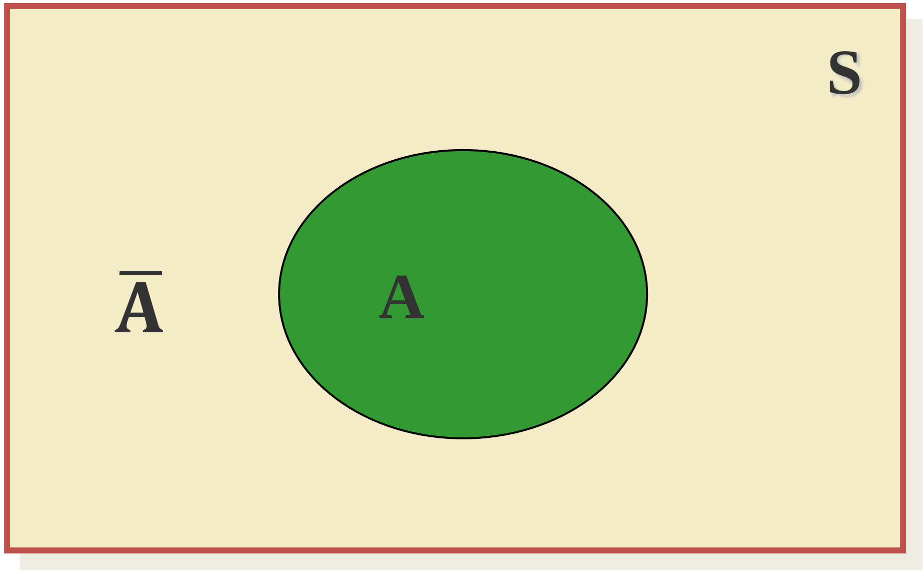
- The <u>intersection</u> of events *A* and *B* is the set of all sample points that are in both *A* and *B*.
- The intersection of events *A* and *B* is denoted by $A \cap B$.



Complement of an Event

- The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event A. We write $\mathbf{A^c}$ **or** $\bar{\mathbf{A}}$.





Complement of an Event

- The <u>complement</u> of event A is defined to be the event consisting of all sample points that are not in A.
- The complement of A is denoted by A^c.

Sample Space S

Venn Diagram

Example

- Select a student from the classroom and record his/her **hair color** and **gender**.
  - **A:** student has brown hair
  - **B:** student is female
  - **C:** student is male

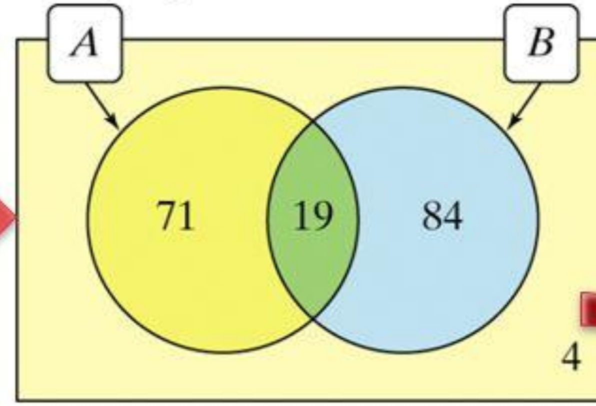
Mutually exclusive; $B = C^C$

- What is the relationship between events **B** and **C**?
- **A^C:** Student does not have brown hair
- **$B \cap C$:** Student is both male and female = $\varnothing$
- **$B \cup C$:** Student is either male and female = all students = S

Venn Diagrams and Probability

Recall the example on gender and pierced ears. We can use a Venn diagram to display the information and determine probabilities.

	Pierced Ears?		
Gender	**Yes**	**No**	**Total**
Male	19	71	**90**
Female	84	4	**88**
Total	**103**	**75**	**178**



Define events *A*: is male and *B*: has pierced ears.

Region in Venn diagram	In words	In symbols	Count
In the intersection of two circles	Male and pierced ears	$A \cap B$	19
Inside circle *A*, outside circle *B*	Male and no pierced ears	$A \cap B^C$	71
Inside circle *B*, outside circle *A*	Female and pierced ears	$A^C \cap B$	84
Outside both circles	Female and no pierced ears	$A^C \cap B^C$	4

Event Relations

- Key words to recognise which event relation you should perform:

1. **Union**: if you see the word **or** or **at least one of** the two event occurs.
2. **Intersection**: if you see the word **and** or **both events** occur.
3. **Complement**: f you see the word **not**.

De Moivre's Laws

1. $\overline{A \cup B} = \bar{A} \cap \bar{B}$

The event A or B does not occur if the event A does not occur and the event B does not occur.

2. $\overline{A \cap B} = \bar{A} \cup \bar{B}$

The event A and B does not occur if the event A does not occur or the event B does not occur.

Rules

- Roles involving the empty set (ϕ) and the entire event (S)

1. $A \cup \phi = A$
2. $A \cap S = A$
3. $A \cap \phi = \phi$
4. $A \cup S = S$
5. $A \cap \bar{A} = \phi$
6. $A \cup \bar{A} = S$

Events and Their Probabilities

An event is a collection of sample points.

The probability of any event is equal to the sum of the probabilities of the sample points in the event.

If we can identify all the sample points of an experiment and assign a probability to each, we can compute the probability of an event.

Assigning Probabilities to Events

Probability of an event P(E): "Chance" that an event will occur

- Must lie between 0 and 1
- "0" implies that the event will not occur
- "1" implies that the event will occur

Types of Probability:

- **Objective**
  - ✓ Relative Frequency Approach
  - ✓ Equally-likely Approach
- **Subjective**

Relative Frequency Approach: Relative frequency of an event occurring in an infinitely large number of trials

Time Period	Number of Male Live Births	Total Number of Live Births	Relative Frequency of Live Male Birth
1965	1927.054	3760.358	0.51247
1965-1969	9219.202	17989.360	0.51248
1965-1974	17857.860	34832.050	0.51268

Equally-likely Approach: If an experiment must result in n equally likely outcomes, then each possible outcome must have probability 1/n of occurring.

Examples:

1. Roll a fair die
2. Select a SRS of size 2 from a population

Subjective Probability: A number between 0 and 1 that reflects a person's degree of belief that an event will occur

Example: Predictions for rain

Assigning Probabilities

Classical Method

Assigning probabilities based on the assumption of equally likely outcomes

Relative Frequency Method

Assigning probabilities based on experimentation or historical data

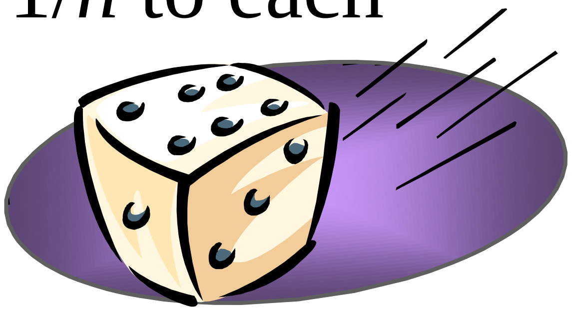
Subjective Method

Assigning probabilities based on judgment

Classical Method

If an experiment has *n* possible outcomes, this method

would assign a probability of 1/*n* to each outcome.



- Example
- Experiment: Rolling a die
- Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

Probabilities: Each sample point has a 1/6 chance of occurring

Relative Frequency Method

Each probability assignment is given by dividing the frequency (number of days) by the total frequency (total number of days).

Number of Polishers Rented	Number of Days	Probability
0	4	.10
1	6	.15
2	18	.45
3	10	.25
4	2	.05
	40	1.00

4/40

Subjective Method

Applying the subjective method, an analyst made the following probability assignments.

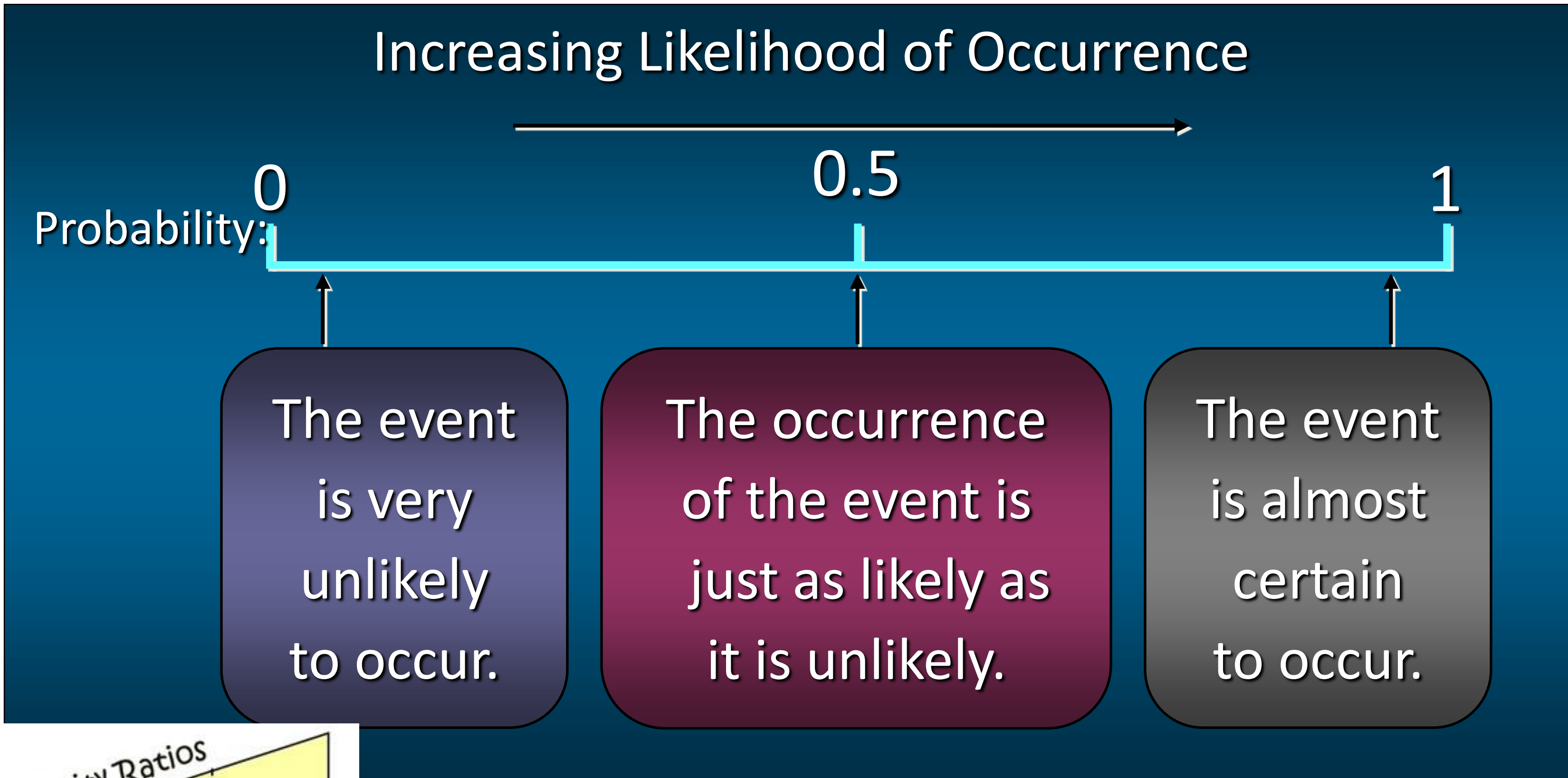
Exper. Outcome	Net Gain or Loss	Probability
(10, 8)	$18,000 Gain	.20
(10, –2)	$8,000 Gain	.08
(5, 8)	$13,000 Gain	.16
(5, –2)	$3,000 Gain	.26
(0, 8)	$8,000 Gain	.10
(0, –2)	$2,000 Loss	.12
(–20, 8)	$12,000 Loss	.02
(–20, –2)	$22,000 Loss	.06

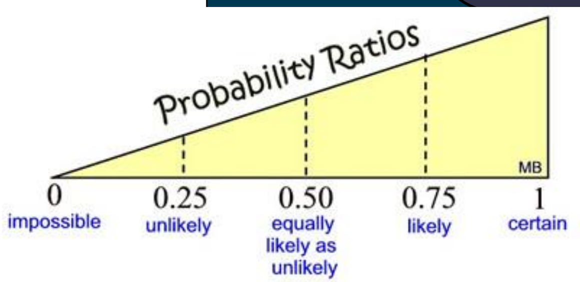
The Probability of an Event

- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0. If event A always occurs when the experiment is performed, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1.

•The **probability of an event A** is found by adding the probabilities of all the simple events contained in A.

Probability as a Numerical Measure of the Likelihood of Occurrence





The Probability of an Event

- The probability of an event A measures "how often" we think A will occur. We write **P(A).**
- Suppose that an experiment is performed *n* times. The relative frequency for an event A is

$$\frac{\text{Number of times A occurs}}{n} = \frac{f}{n}$$

Let A be the event A = $\{o_1, o_2, \ldots, o_k\}$, where $o_1, o_2, \ldots, o_k$ are k different outcomes. Then

$$P(A) = P(o_1) + P(o_2) + \cdots + P(o_k)$$

Finding Probabilities

- Probabilities can be found using
  – Estimates from empirical studies
  – Common sense estimates based on equally likely events.

- **Examples:**
  – Toss a fair coin. P(Head) = 1/2
  – 10% of the U.S. population has red hair. Select a person at random. P(Red hair) = .10

Example

- Toss two coins. What is the probability of observing at least one head?

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	1/4
	T	HT	1/4
T	H	TH	1/4
	T	TT	1/4

P(at least 1 head)

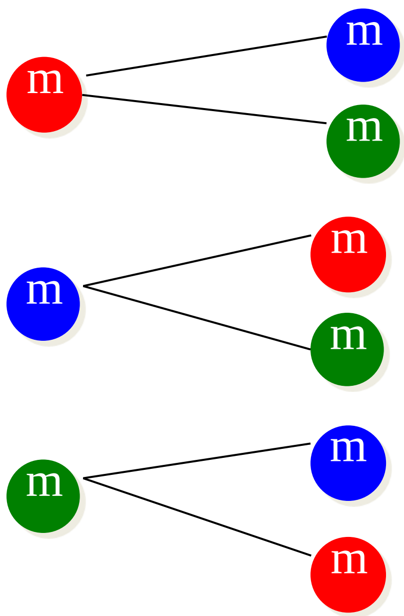
$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

Example

- A bowl contains three M&Ms®, one red, one blue and one green. A child selects two M&Ms at random. What is the probability that at least one is red?

1st M&M 2nd M&M



E_i	$P(E_i)$
RB	1/6
RG	1/6
BR	1/6
BG	1/6
GB	1/6
GR	1/6

P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

Counting Rules

- If the simple events in an experiment are **equally likely,** you can calculate

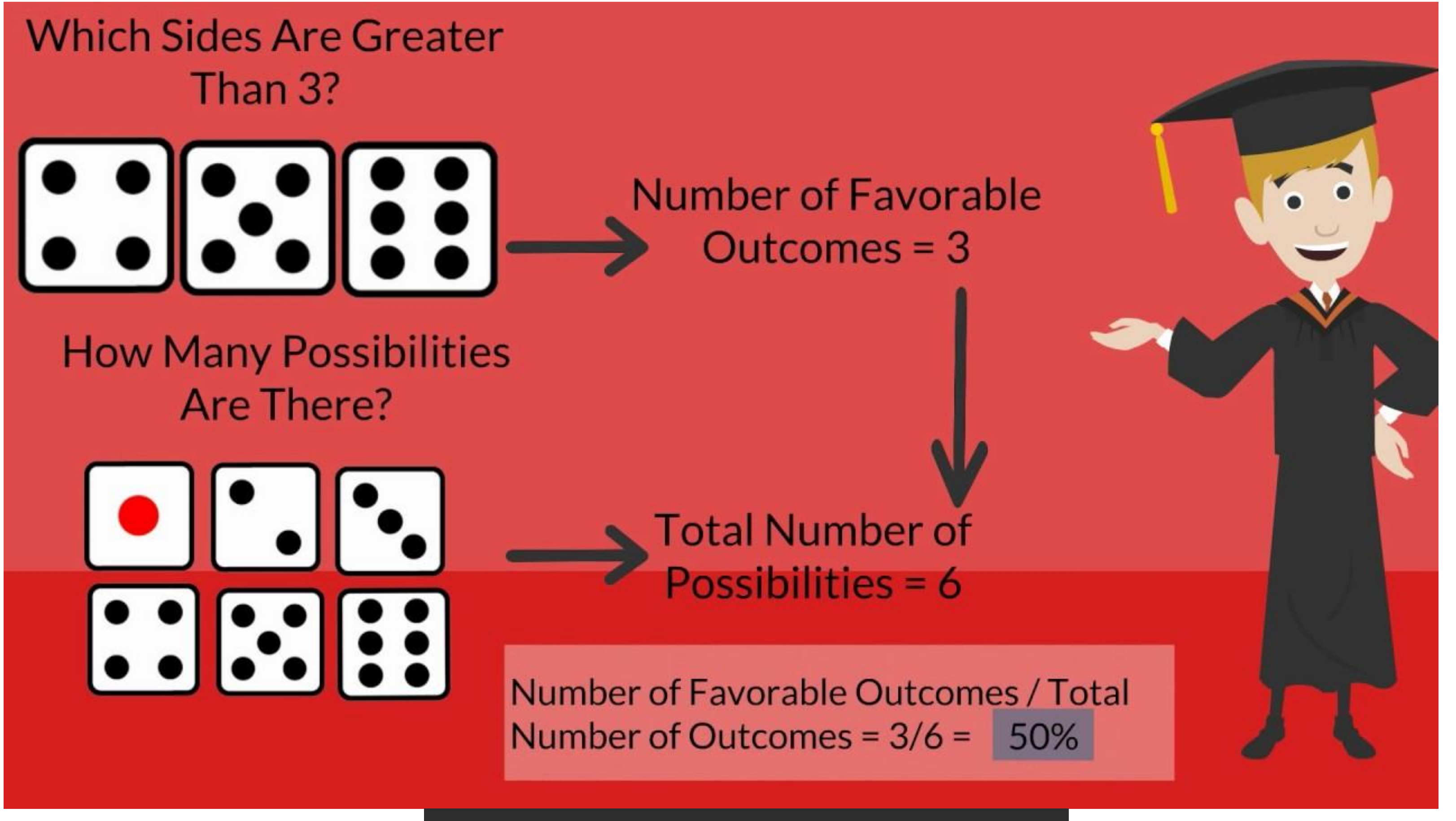
$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

- You can use **counting rules** to find n_A and N.

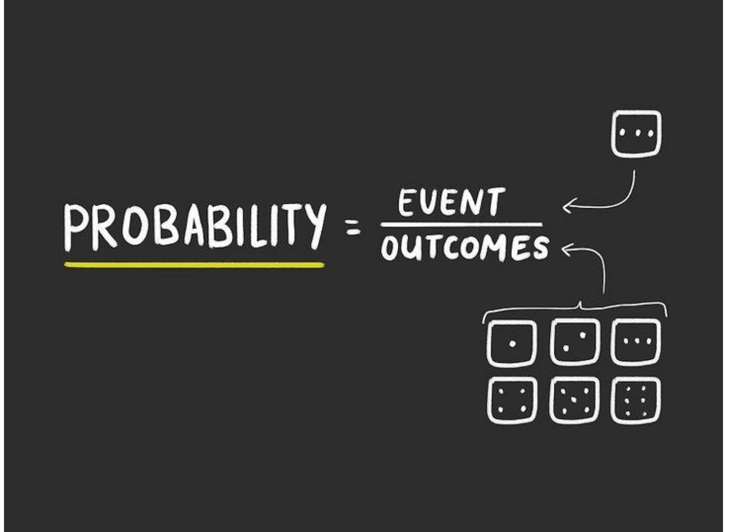
A Counting Rule for Multiple-Step Experiments

If an experiment consists of a sequence of k steps in which there are n_1 possible results for the first step, n_2 possible results for the second step, and so on, then the total number of experimental outcomes is given by $(n_1)(n_2) \ldots (n_k)$.

A helpful graphical representation of a multiple-step experiment is a tree diagram.



Which Sides Are Greater Than 3?
Number of Favorable Outcomes = 3
How Many Possibilities Are There?
Total Number of Possibilities = 6
Number of Favorable Outcomes / Total Number of Outcomes = 3/6 = 50%



PROBABILITY = EVENT / OUTCOMES

Counting Rule for Permutations

A second useful counting rule enables us to count the number of experimental outcomes when n objects are to be selected from a set of N objects, where the order of selection is important.

- Number of Permutations of N Objects Taken n at a Time

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!}$$

where: $N! = N(N-1)(N-2)\ldots(2)(1)$

$n! = n(n-1)(n-2)\ldots(2)(1)$

$0! = 1$

Permutations

- The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

where $n! = n(n-1)(n-2)...(2)(1)$ and $0! \equiv 1.$

Example: How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

The order of the choice is important!

$$P_3^4 = \frac{4!}{1!} = 4(3)(2) = 24$$

Examples

Example: A lock consists of five parts and can be assembled in any order. A quality control engineer wants to test each order for efficiency of assembly. How many orders are there?

The order of the choice is important!

$$P_5^5 = \frac{5!}{0!} = 5(4)(3)(2)(1) = 120$$

Counting Rule for Combinations

A third useful counting rule enables us to count the number of experimental outcomes when n objects are to be selected from a set of N objects.

Number of Combinations of N Objects Taken n at a Time

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where: $N! = N(N-1)(N-2) \ldots (2)(1)$

$n! = n(n-1)(n-2) \ldots (2)(1)$

$0! = 1$

Combinations

- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is

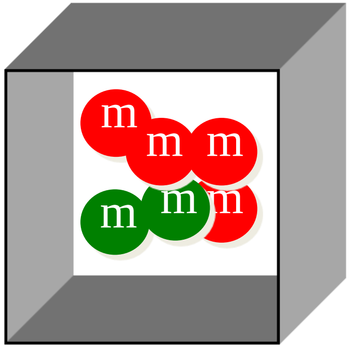
$$C_r^n = \frac{n!}{r!(n-r)!}$$

Example: Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

Example



- A box contains six M&Ms®, four red and two green. A child selects two M&Ms at random. What is the probability that exactly one is red?

The order of the choice is not important!

$$C_2^6 = \frac{6!}{2!4!} = \frac{6(5)}{2(1)} = 15$$

ways to choose 2 M&Ms.

$$C_1^2 = \frac{2!}{1!1!} = 2$$

ways to choose 1 green M&M.

$$C_1^4 = \frac{4!}{1!3!} = 4$$

ways to choose 1 red M&M.

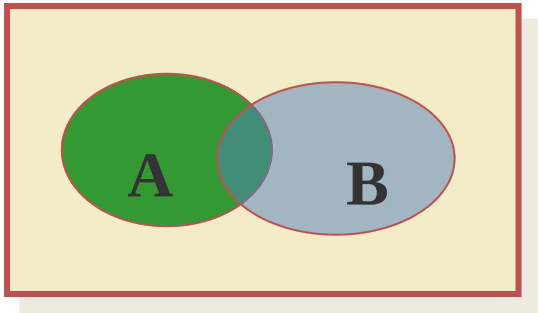
$4 \times 2 = 8$ ways to choose 1 red and 1 green M&M.

P(exactly one red) = 8/15

Calculating Probabilities for Unions and Complements

- There are special rules that will allow you to calculate probabilities for composite events.
- **The Additive Rule for Unions:**
- For any two events, **A** and **B**, the probability of their union, **P(A ∪ B)**, is

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$



Addition Law

- The addition law provides a way to compute the probability of event *A*, or *B*, or both *A* and *B* occurring.

- The law is written as:

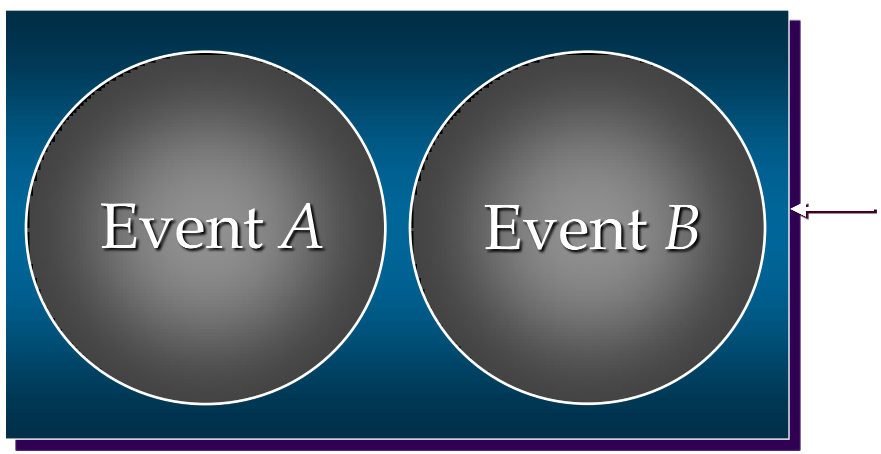
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Mutually Exclusive (Disjoint) Events

Two events are said to be <u>mutually exclusive</u> if the events have no sample points (outcomes) in common.

Two events are mutually exclusive if, when one event occurs, the other cannot occur (They can't occur at the same time. The outcome of the random experiment cannot belong to both A and B

If events A and B are mutually exclusive, $P(A \cap B) = 0$.



Sample Space *S*

Mutually Exclusive Events

- If events A and B are mutually exclusive, $P(A \cap B) = 0$.
- The addition law for mutually exclusive events is:

$$P(A \cup B) = P(A) + P(B)$$

there's no need to include "– $P(A \cap B)$"

Mutually Exclusive (Disjoint) Events

- **Two events are mutually exclusive if, when one event occurs, the other cannot, and vice versa.**

- Experiment: Toss a die
  - A: observe an odd number
  - B: observe a number greater than 2
  - C: observe a 6
  - D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

B and C?
B and D?

Example: Additive Rule

Example: Suppose that there were 120 students in the classroom, and that they could be classified as follows:

A: brown hair

P(A) = 50/120

B: female

P(B) = 60/120

	Brown	**Not Brown**
Male	**20**	**40**
Female	**30**	**30**

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 50/120 + 60/120 - 30/120$

$= 80/120 = 2/3$

A Special Case

When two events A and B are **mutually exclusive,** $P(A \cap B) = 0$ and $\mathbf{P(A \cup B) = P(A) + P(B)}$.

A: male with brown hair

$P(A) = 20/120$

B: female with brown hair

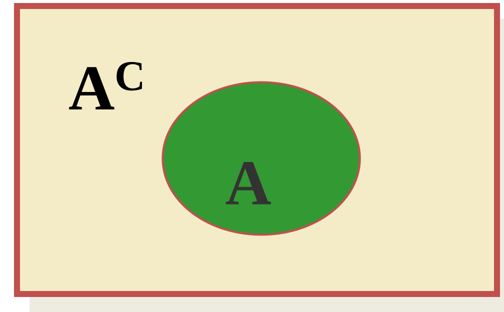
$P(B) = 30/120$

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are mutually exclusive, so that →

$P(A \cup B) = P(A) + P(B)$
$= 20/120 + 30/120$
$= 50/120$

Calculating Probabilities for Complements



- We know that for any event **A:**

  $P(A \cap A^C) = 0$

- Since either **A** or **A**^{C} must occur,

  $P(A \cup A^C) = 1$

- so that $P(A \cup A^C) = P(A) + P(A^C) = 1$

$$P(A^C) = 1 - P(A)$$

Example

Select a student at random from the classroom. Define:

A: male

P(A) = 60/120

B: female

	Brown	Not Brown
Male	20	40
Female	30	30

A and B are complementary, so that

P(B) = 1- P(A)
= 1- 60/120 = 40/120

Calculating Probabilities for Intersections

- we can find $P(A \cap B)$ directly from the table. Sometimes this is impractical or impossible. The rule for calculating $P(A \cap B)$ depends on the idea of **independent and dependent events.**

Two events, **A** and **B**, are said to be **independent** if and only if the probability that event **A** occurs does not change, depending on whether or not event **B** has occurred.

Conditional Probabilities

- The probability that A occurs, given that event B has occurred is called the **conditional probability** of A given B and is defined as

$$P(A|B)=\frac{P(A\cap B)}{P(B)}\text{ if }P(B)\neq 0$$

"given"

Conditional Probability

- The probability of an event given that another event has occurred is called a conditional probability.
- The conditional probability of A given B is denoted by $P(A \mid B)$.
- A conditional probability is computed as follows :

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Multiplication Law

- The multiplication law provides a way to compute the probability of the intersection of two events.
- The law is written as:

$$P(A \cap B) = P(B)P(A \mid B)$$

Example 1

- Toss a fair coin twice. Define
  - A: head on second toss
  - B: head on first toss

HH	1/4
HT	1/4
TH	1/4
TT	1/4

$P(A|B) = ½$

$P(A|\text{not } B) = ½$

P(A) does not change, whether B happens or not…

→ A and B are independent!

Independent Events

- If the probability of event *A* is not changed by the existence of event *B*, we would say that events *A* and *B* are <u>independent</u>.

- Two events *A* and *B* are independent if:

  $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

P(A) does change, depending on whether B happens or not… → A and B are dependent!

Defining Independence

- We can redefine independence in terms of conditional probabilities:

Two events A and B are **independent** if and only if

$$P(A|B) = P(A) \quad \text{or} \quad P(B|A) = P(B)$$

Otherwise, they are **dependent**.

- Once you've decided whether or not two events are independent, you can use the following rule to calculate their intersection.

Multiplication Law for Independent Events

- The multiplication law also can be used as a test to see if two events are independent.
- The law is written as:

$$P(A \cap B) = P(A)P(B)$$

The Multiplicative Rule for Intersections

- For any two events, **A** and **B**, the probability that both **A** and **B** occur is

$$P(A \cap B) = P(A)\, P(B \text{ given that } A \text{ occurred}) = P(A)P(B|A)$$

- If the events **A** and **B** are independent, then the probability that both **A** and **B** occur is

$$P(A \cap B) = P(A)\, P(B)$$

Example 1

In a certain population, 10% of the people can be classified as being high risk for a heart attack. Three people are randomly selected from this population. What is the probability that exactly one of the three are high risk?

Define H: high risk N: not high risk

P(exactly one high risk) = P(HNN) + P(NHN) + P(NNH)

= P(H)P(N)P(N) + P(N)P(H)P(N) + P(N)P(N)P(H)

$= (.1)(.9)(.9) + (.9)(.1)(.9) + (.9)(.9)(.1) = 3(.1)(.9)^2 = .243$

Example 2

Suppose we have additional information in the previous example. We know that only 49% of the population are female. Also, of the female patients, 8% are high risk. A single person is selected at random. What is the probability that it is a high risk female?

Define H: high risk F: female

From the example, P(F) = .49 and P(H|F) = .08.
Use the Multiplicative Rule:

$$P(\text{high risk female}) = P(H \cap F)$$

$$= P(F)P(H|F) = .49(.08) = .0392$$

Rules Summary

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cup B) = P(A) + P(B)$
- $P(A^C) = 1 - P(A)$
- $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$
- $P(A \cap B) = P(B).\, P(A/B)$
- $P(A/B) = P(A)$
- $P(B/A) = P(B)$
- $P(A \cap B) = P(B).P(A)$

- Additional rules:
  - $P(A \cap \bar{B}) = P(A) - P(A \cap B)$
  - $P(\bar{A} \cap B) = P(B) - P(A \cap B)$
  - $P(A \cup \bar{B}) = P(\bar{B}) + P(A \cap B)$
  - $P(\bar{A} \cup B) = P(\bar{A}) + P(A \cap B)$
  - De Morgan's Laws
  - $P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$
  - $P((\overline{A \cap B}) = P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$

Examples

For the random experiment of rolling two fair dice, suppose that we define the two events A and B as follows:

A: First die show 1

B: Second die show 1

Then find the value of $P(A \cup B)$?

Solution

The outcomes of the two events are as follows:

$$A = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\} \Rightarrow P(A) = 6/36$$

$$B = \{(1,1), (2,1), (3,1), (4,1), (5,1), (6,1)\} \Rightarrow P(B) = 6/36$$

$$A \cap B = \{(1,1)\} \Rightarrow P(A \cap B) = 1/36$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 6/36 + 6/36 - 1/36$$

$$= 11/36$$

$$= 0.306$$

Example-Disjoint Events

- For random experiment of rolling two fair dice, suppose that we define the two events A and B as follows:

A: Sum of the two numbers comes up to 3

B: Sum of the two numbers comes up to 6

Then find $P(A \cup B)$?

$A = \{ (1,2), (2,1) \}$ $P(A) = 2/36$

$B = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$ $P(B) = 5/36$

$P(A \cap B) = \phi$

So $P(A \cup B) = (2/36) + (5/36) = (7/36) = 0.194$

Examples

- Suppose that in the Gulf pharmaceutical industries company in UAE, there are two telephone lines 1 and II. Let A be the event that line I is busy and let B be the event that line II is busy.

$P(A)=0.55$ $P(B)=0.65$ and $P(A \cap B)=0.35$

Answer the following:

a) Find the probability that both lines are free?

Examples

Solution

P(Both lines are free) = $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$

$= 1 - P(A \cup B)$

$= 1 - (P(A) + P(B) - P(A \cap B)$

$= 1 - (0.55 + 0.65 - 0.35)$

$= 1 - 0.85$

$= 0.15$

Continued

b) Find the probability that line I is busy and line II is free?

Solution

P(Line I is busy and line II is free)=$P(A\cap \overline{B})$

$=P(A\cap \overline{B})=P(A)-P(A\cap B)$

$=0.55 - 0.35$

$=0.20$

Continued

c) Find the probability that line I is free or line II is busy?

Solution

P(line I is free or line II is busy)=P($\bar{A}$ ∪ B)

= P($\bar{A}$)+ P(A∩B)

=1-P(A)+ P(A∩B)

=1-(0.55+0.35)

= 0.8

Additional Rules

- $P(A/\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$

$P(\bar{A}/B) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(B) - P(P \cap B)}{P(B)} = 1 - P(A/B)$

$P(\bar{A}/\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{P(\overline{A \cup B})}{1 - P(B)} = \frac{1 - P(A \cup B)}{1 - P(B)} = \frac{1 - [P(A) + P(B) - P(A \cap B)}{1 - P(B)}$

Example

For the random experiment of rolling two fair dice, suppose that we define the two events A and B as follows:

A: First die shows 1

B: Second die show 2

Then find the value of P(A/B)

Continued

Solution

- P(A/B) = $\frac{P(A \cap B)}{P(B)}$
- A={(1,1), (1,2), (1,3), (1,4),(1,5), (1,6)}
  B={(1,2), (2,2), (3,2), (4,2), (5,2), (6,2)}
  A∩B = {(1,2)} = P(A∩B) = 1/36

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{1/36}{6/36}$$

$$= \frac{1}{6}$$

$$= 0.167$$

Example

In the Queen Alia International Airport, suppose that the probability that a regularly scheduled flight departs on time is P(A) = 0.83 ; the probability that a regularly scheduled flight arrives on time is P(B) = 0.92 ; and the probability that it departs and arrives on time is $P(A \cap B) = 0.78$. Find the probability that a plane

a) Arrives on time given that it departed on time, and

b) Departed on time given that it has arrived on time?

Solution

a) $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{0.78}{0.83} = 0.94$

b) $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.78}{0.92} = 0.85$

Example

Suppose that we have a drug box containing 20 tablets, of which 5 are defective, if 2 tablets are selected at random and removed from the box in succession without replacing the first, what is the probability that both tablets are defective?

Solution

We shall let:

A: the event that the first tablet is defective

B: the event that the second tablet is defective

$A \cap B$: the event that A occurs, and then B occurs after A has occurred (both events occur)

Continued

- The probability of first removing a defective tablet is 5/20 = ¼, then the probability of removing a second defective tablet from the remaining 4 is 4/19. hence:
- $P(A \cap B) = P(B).\ P(A/B)$

$$= (1/4).\ (4/19)$$

$$= 1/19 = 0.053$$

Additional Rules

Definition of Independence

Two events A and B are said to be (statistically) independent if any one of the following equivalent conditions holds:

1) $P(A \cap B) = P(A)P(B)$
2) $P(A/B) = P(A)$
3) $P(B/A) = P(B)$

Otherwise, they are said to be dependent events

Then,

1) A and not B are independent, that is: $P(A \cap \bar{B}) = P(A)P(\bar{B})$
2) Not A and B are independent, that is: $P(\bar{A} \cap B) = P(\bar{A})P(B)$
3) Not A and not B are independent, that is: $P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B})$

Example

Two dice are rolled. Suppose that we define the two events A and B as follows:

A: First die show 1.

B: The sum of the two numbers comes up on the two dice is 7.

Are the two events A and B independent?

Solution

A = {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)}

B = {(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)}

$A \cap B = \{(1,6)\}$

(1) $P(A) = (\frac{6}{36})$ *and* $P(B) = (\frac{6}{36})$

then $P(A)P(B) = \frac{1}{36}$

(2) $P(A \cap B) = \frac{1}{36}$

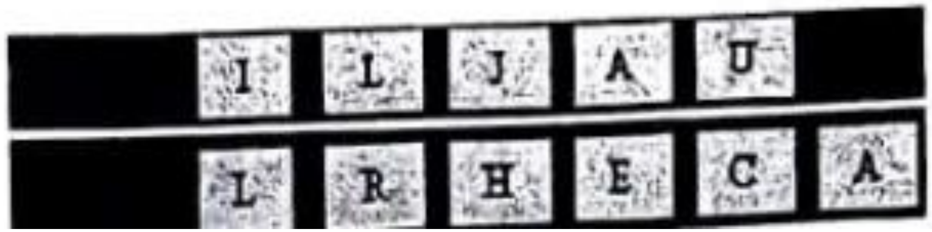
Then we get that $P(A \cap B) = \frac{1}{36} = P(A)P(B)$

Example

Two sets of cards with a letter on each card as follows are placed into separate bags :

Bag Number I

Bag Number II



Sara randomly picked one card from each bag. Find the probability that:

a) She picked the letter J and R?

b) Both letters are L?

c) Both letters are vowels?

Solution

(a) Probability that she picked J and R

= (1/5)(1/6) = 1/30 = 0.033

(b) Probability that both letters are L

= (1/5)(1/6) = 1/30 = 0.033

(c) Probability that both letters are vowels

= (3/5)(2/6) = 6/30 = 0.2

Contingency Table

A contingency table provides a different way of calculating probabilities. The table helps in determining conditional probabilities quite easily. The table displays sample values in relation to two different variables that may be dependent or contingent on one another. The two variables are divided into several categories with their frequencies.

Example

- A random sample of size 200 adults are classified below according to the sex and the level of education attained:

Education	Male (M)	Female (F)	Total
Elementary (E)	38	45	83
Secondary (S)	28	50	78
Higher (H)	22	17	39
Total	88	112	200

Continued

Suppose that a person is selected at random from this group, then find the probability that:

(a) the person is male?

Solution

P(the person is male) = P(M) = 88 / 200 = 0.44

(b) the person has a secondary education?

Solution

P(the person has a secondary education)

= P(S) = 78 / 200 = 0.39

(c) the person is not a male?

Solution

P(the person is not a male) = 1- P(the person is male)

= 1 - P(M) = 1 – 0.44 = 0.56

OR

P(the person is not a male) = P(the person is female)

= P(F) = 112 / 200 =

Continued

d) The person is a male or has higher education?

Solution:

P(the person is a male or has higher education)

$P(M \cup H)=P(M)+P(H)-P(M \cap H)=(88/200)+(39/200)-(22/200)=105/200=0.525$

e) The person is female and not has an elementary education

Solution

P(the person is a female and not has an elementary education)

$P(F \cap \overline{E}) = P(F)-P(F \cap E)= (112/200)-(45/200)=67/200=0.335$

f) The person is a male or has a secondary education?

P(the person is not a female or has a secondary education)

$P(\overline{F} \cup S)=P(\overline{F})+P(F \cap S)=(88/200)+(50/200)=(138/200)=0.69$

Continued

g) The person is neither a male nor has a higher education?

solution#

P(the person is neither a male nor has a higher education)

$=P(\bar{M} \cap \bar{H})$

$=(P(\overline{M \cup H})$

$=1-P(M\cup H)$

$=1-[P(M)+P(H)-P(M\cap H)$

$=1-[88/200+39/200-22/200]=1-(105/200)= 0.475$

h) The person is a male given that the person has a secondary education?

Solution

P(the person is a male given that the person has a secondary education)

$P(M/S) =P(M\cap S)/P(S)=(28/200)/(78/200)=28/78=0.359$

Continued

i) The person does not have a higher education degree given that the person is a female?

Solution

P(the person does not have a higher education degree given that the person is a female)

$$P(\overline{H}/F) = \frac{P(\bar{H} \cap F)}{P(F)} = \frac{P(F) - P(H \cap F)}{P(F)} = 1 - P(H/F) =$$

$$= 1 - \frac{P(H \cap F)}{P(F)} = 1 - ((17/200)/(112/200)) = 1 - (17/112) = 0.848$$